

Multi-Attribute Profit-Maximizing Pricing

Extended Abstract*

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Abstract

In the *unlimited-supply profit-maximizing pricing* problem, we are given the consumers' consideration sets and know their purchase strategy (e.g. buy the cheapest items). The goal is to price the items to maximize the revenue. Previous studies suggest that this problem is too general to obtain even a sublinear approximation ratio (in terms of the number of items) even when the consumers are restricted to have very simple purchase strategies.

In this paper we initiate the study of the *multi-attribute pricing* problem as a direction to break this barrier. Specifically, we consider the case where each item has a constant number of *attributes*, and each consumer would like to buy the items that satisfy her *criteria* in all attributes. This notion intuitively captures typical real-world settings and has been widely-studied in marketing research, healthcare economics, etc. It also helps categorizing previously studied cases, such as *highway pricing* problem and *graph vertex pricing* problem on planar and bipartite graphs, from the general case.

We show that this notion of attributes leads to improved approximation ratios on a large class of problems. This is obtained by utilizing the fact that the consideration sets have low VC-dimension and applying Dilworth's theorem on a certain partial order defined on the set of items. As a consequence, we present sublinear-approximation algorithms, thus breaking the previous barrier, for two well-known variants of the problem: *unit-demand uniform-budget min-buying* and *single-minded* pricing problems. Moreover, we generalize these techniques to the *unit-demand utility-maximizing* pricing problem and (non-uniform) unit-demand min-buying pricing problem when valuations or budgets depend on attributes, as well as the pricing problem with *symmetric valuations* and *subadditive revenues*. These results suggest that considering attributes is a promising research direction in obtaining improved approximation algorithms for such pricing problems.

*This extended abstract focuses on explaining the main technical idea by showing a sub-linear approximation algorithm for the problem called UUDP-MIN. For the result in its full generality, we refer to the full version [17].

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1 Introduction

This paper studies one variation of the *unlimited-supply pricing* problem in various decision models. To keep our discussion simple, let us consider the following problem as an example. We are given a set \mathcal{I} of n consumers and a set \mathcal{C} of m items. Every item $\mathbf{I} \in \mathcal{I}$ is represented by a point $\mathbf{I} = (\mathbf{I}[1], \dots, \mathbf{I}[d]) \in \mathbb{R}_{\geq 0}^d$, where $\mathbb{R}_{\geq 0}$ denotes the set of non-negative reals and $\mathbf{I}[j]$ expresses the quality of item \mathbf{I} in the j -th attribute. Every consumer $\mathbf{C} \in \mathcal{C}$ is represented by a point $\mathbf{C} = (\mathbf{C}[1], \dots, \mathbf{C}[d]) \in \mathbb{R}_{\geq 0}^d$, where $\mathbf{C}[j]$ is the criterion of consumer $\mathbf{C} \in \mathcal{C}$ in the j -th attribute. Each consumer \mathbf{C} is additionally equipped with budget $B_{\mathbf{C}} \in \mathbb{R}_{\geq 0}$ and a consideration set

$$S_{\mathbf{C}} = \{\mathbf{I} : \mathbf{I}[j] \geq \mathbf{C}[j], \text{ for all } 1 \leq j \leq d\}. \quad (1)$$

Once we assign prices to items, each consumer \mathbf{C} will buy the cheapest item \mathbf{I} in $S_{\mathbf{C}}$ if the price of item \mathbf{I} is at most $B_{\mathbf{C}}$. The objective is to set the price of items in \mathcal{I} in order to maximize the revenue. We call this problem *d-attribute uniform-budget unit-demand min-buying pricing problem* (*d-UUDP-MIN*). To illustrate the geometric flavor of the problem, Fig. 1 shows the case of $d = 2$ as an example. Here, each item corresponds to a point in the plane. The consideration set of each consumer \mathbf{C} is represented by an (unbounded) axis-parallel rectangle with point \mathbf{C} as a lower-left corner.

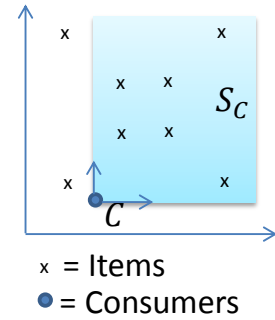


Figure 1: Problem visualization

Motivation The *d-UUDP-MIN* problem above is in fact a special case of the well-known *uniform-budget unit-demand min-buying pricing problem* (*UUDP-MIN*) which has been widely studied recently (see, e.g., [46, 35, 47, 14] and its generalization [1]). However, instead of formulating the consideration set $S_{\mathbf{C}}$ as in Eq. (1), consideration sets $S_{\mathbf{C}}$ in the *UUDP-MIN* problem have no specific form and have to be specified explicitly. This problem is known to be $O(\log m)$ -approximable [1], so we have a reasonable approximation guarantee when there are not many consumers¹. However, in many cases, one would expect the number of consumers to be much larger than the number of items n . In this case, we would be stuck at $O(n)$ approximation ratio. Moreover, there is evidence to suggest that improving this might be impossible: under some reasonable complexity assumptions [28], the problem is hard to approximate within a factor of $O(\log^\epsilon m)$ and $O(n^\epsilon)$ for some $\epsilon > 0$ [13, 14, 23].

This calls for more insights from real-world scenarios to help overcome this barrier. Two previous research directions along this line are to exploit either input distributions or special structures. This paper focuses on making a step further in the second direction. In this direction, we have seen some progress in recent years (e.g., when there is a *price-ladder constraint* [46, 1, 47, 48, 15], consideration sets are small [5, 15] or consideration sets correspond to paths on graphs [5, 26, 34, 25, 30]). However, the previously studied structures are very specific to special scenarios, and thus it has been a challenging task to find simplified problem structures that capture various real-world settings while allowing us to break the previous barrier.

¹In fact, this approximation guarantee holds even for the *general pricing problem* defined in the full version of this paper (using [7]).

1.1 Our Approach & Results

To break through the approximation barrier of the UUDP-MIN problem and at the same time capture a broad class of real-world settings, we propose a natural structure based on the following simple observation on the consumers' behavior. Consider a setting where we sell cars. If a consumer has car A with horse power 130HP in her consideration set, she would not mind buying car B with horse power 150HP. Maybe she does not want B because it is less energy-efficient or has lower reputation. But, if we list *all* attributes of the cars that people care about and it happens that B is not worse than A in all other aspects, then B should also be in the list.

In particular, instead of looking at a full generality where each consumer \mathbf{C} considers any set of items $S_{\mathbf{C}}$, it is reasonable to assume that each consumer has some criterion in mind for each attribute of the cars, and her consideration set consists of any car that passes all her criteria, i.e. consumers judge items according to their attributes. This natural assumption has been a model of study in other fields such as marketing research, healthcare economics and urban planning. It is referred to as the *attribute-based screening process*. In particular, using criteria to define consideration sets as in Eq. (1) is called *conjunctive screening rule*. Besides being natural, this assumption has been supported by a number of studies where it is concluded that consumers typically use a conjunctive screening rule in obtaining their consideration sets (see further detail in Section 1.2).

While the UUDP-MIN problem is likely to be hard to approximate in the general cases, we obtain an $o(n)$ -approximation algorithm for the d -UUDP-MIN problem. Our main result is summarized as follows.

Theorem 1.1. *For any constant d , there is an $\tilde{O}_d(n^{1-\epsilon(d)})$ approximation algorithm for d -UUDP-MIN problem where function $\epsilon(d) := 1/4^{d-1}$ and \tilde{O}_d treats d as a constant and hides a $\text{polylog}(n)$ factor.*

The essential idea behind our algorithm is to partition the problem instance into sub-instances without decreasing the optimal revenue (we call this *consideration-preserving decomposition*). This is done by using Dilworth's Theorem (partitioning items into chains and anti-chains) and epsilon-nets to find subsets of items satisfying certain structural properties. Subsequently, we show that the dimensions of these sub-instances can be reduced through the notion of *consideration-preserving embedding*. In the end of our algorithm, we are left with a sub-linear number of sub-instances, each of which can be solved almost optimally in polynomial time. Returning the best solution among the solutions of these sub-instances guarantees a sub-linear approximation ratio. We note that the value of $\epsilon(d)$ could be improved slightly to $1/4^{d-2}$ since 2-UUDP-MIN admits Quasi-Polynomial Time Approximation Schemes (QPTAS) (see Appendix B).

Further results We note that the notion of attribute and our techniques are useful not only for attacking UUDP-MIN but also for several other models. For instance we consider another well-known setting called *single-minded pricing* (SMP) [35], in which each consumer buys the entire bundle $S_{\mathbf{C}}$ if she can afford to and buys nothing otherwise. We prove that the d -attribute version of SMP, called d -SMP, admits the same approximation ratio as in Theorem 1.1. Although d -SMP is not as applicable as d -UUDP-MIN, it is interesting that d -SMP captures many previously studied problems as special cases. For example, 2-SMP generalizes the highway pricing problem [35, 5, 26, 34] and thus our algorithmic results on 2-SMP can also be applied to this problem. Moreover, 3-SMP generalizes the upward case of the tollbooth pricing problem [25, 42] as well as the graph vertex pricing problem on planar graphs [5, 18]. Finally, 4-SMP generalizes the unlimited-supply version

of the *exhibition* problem [22], as well as the graph vertex pricing problem on bipartite graphs [5, 41].

More importantly, we obtain QPTASs for important subroutines which are crucial building blocks in our main theorem. These results, together with a widely-believed assumption that the existence of a QPTAS for any problem implies that PTAS exists for the same problem (e.g., [8, 26]), imply that the approximation ratios of the aforementioned problems can be improved. (We emphasize that this claim relies on the fact that the subroutines that we obtain QPTASs have PTASs.) They also imply that obtaining sublinear-approximation algorithms is possible even for more general problems. Here, we consider two broader models of the pricing problem. Due to the page limitation, most of the proof details are omitted in this extended abstract and can be found in the full version of our paper [17].

- The *unit-demand utility-maximizing pricing problem* (UDP-Util) is a model that generalizes UUDP-MIN in that the consumers may have different *valuations* on different items and want to maximize the difference between the valuations and the price of the purchased items. (This problem is sometimes called *envy-free pricing problem* [15] and *max-gain-buying pricing problem* [1]. We name this problem UDP-Util to distinguish it from other general pricing problems studied in this paper.) We show that there is an $\tilde{O}_d\left(n^{1-1/4^{d-1}}\right)$ -approximation algorithm for a naturally defined d -attribute version of UDP-Util when valuations are monotone functions of d attributes (i.e., each consumer values an item not less than the inferior ones). This result also holds for the d -attribute version of the (non-uniform) unit-demand min-buying pricing problem (UDP-MIN) where consumers can have different budgets on different items.
- The pricing problem with *symmetric valuations* and *subadditive revenues* is a model that includes both UUDP-MIN and SMP as well as many other natural problems. Informally, symmetric valuation means that each consumer's valuation only depends on the number of items she gets, and subadditive revenue implies that the amount that each consumer pays to get an item bundle X is at most the amount she pays to get the items in X separately. We show that there is an $\tilde{O}_d\left(n^{1-1/4^{d-1}}\right)$ -approximation algorithm for the d -attribute version of this problem.

As a by-product of our proof, we show a QPTAS for 2-SMP which subsumes the previous QPTAS for the highway pricing problem [26].

Hardness We also study the hardness of approximation of our problems. We show that 3-UUDP-MIN and 2-SMP are NP-hard, and 4-UUDP-MIN and 4-SMP are APX-hard. Hence, our problem is already non-trivial for small d . Our hardness proofs establish a cute connection between our problem and the vertex cover problem on graphs of low *order dimensions* [49, 50]. Moreover, we show that the hardness of our problem tends to increase as we increase d , and the whole generality is captured when $d = n$. In particular, we show that when the dimension is sufficiently high (i.e. $d \geq \log^2 n$), the problems are hard to approximate to within a factor of d^ϵ for some ϵ . This latter hardness result holds for every model we consider. Table 1 concludes our results for d -UUDP-MIN and d -SMP.

Organization After the end of introduction, the rest of the paper is organized as follows. In Section 2 we give the proof of Theorem 1.1 by showing how to reduce the dimension by one while

Problem		$d = 1$	$d = 2$	$d = 3$	$d = 4$	large d {range }
d -UUDP-MIN	Upper bound	Polytime	QPTAS	NP-hard	APX-hard	$n^{1-\frac{1}{4d-1}}$ {constant d }
	Lower bound					d^ϵ { $d = \omega(\log n)$ }
d -SMP	Upper bound	Polytime	QPTAS		APX-hard	$n^{1-\frac{1}{4d-1}}$ {constant d }
	Lower bound		NP-hard			d^ϵ { $d = \omega(\log n)$ }

Table 1: Results of d -UUDP-MIN and d -SMP for small values of d .

ensuring that the sublinear approximation ratio is preserved (see Theorem 2.1 below). In Section 3, we sketch the key ideas used in proving our hardness results. We conclude with some open problems in Section 4. Other results we mention here can be found in the full version [17].

1.2 Related Work

Rusmevichientong et al. [46, 47, 48] defined the *non-parametric multi-product pricing problem*, motivated by the possibility that the data about consumers’ preferences and budgets can be predicted based on previous data, which can be gathered and mined by web sites designed for this purpose, e.g., [36, 48]. This problem is what we call uniform-budget unit-demand pricing problem (UUDP). Rusmevichientong et al. proposed many decision rules such as min-buying, max-buying and rank-buying and showed that UUDP-MIN allows a polynomial-time algorithm, assuming a *price-ladder constraint*, i.e., a predefined total order on the prices of all products. Aggarwal et al. [1] later showed that the price ladder constraint also leads to a 4-approximation algorithm for the max-buying case, even in the case of limited supply.

We note that the price ladder constraint is closely related to our notion of attributes in the following sense. It can be shown that 1-UUDP-MIN satisfies the price ladder constraint (this is the reason we can solve it in polynomial time). Moreover, although 2-UUDP-MIN does not satisfy this constraint, it *partially* satisfies the constraint in the sense that if one item is better than another item in all attributes then we can assume that it has a higher price. This property plays an important role in obtaining QPTAS for 2-UUDP-MIN and also holds for general d .

Other variants defined later include non-uniform and utility-maximizing unit-demand, single-minded (SMP), tollbooth and highway models [1, 35]. These problems were later found to have important connections to algorithmic mechanism design [2, 6, 35] and online pricing problems [5, 12]. As we mentioned in the introduction, many problems can be approximated within the factor of $O(\log m + \log n)$ and $O(n)$, and these seem to be tight.

Recent research that incorporates more insights from real-world scenarios can be categorized into two directions. The first direction is to assume that consumer valuations are drawn from some known distributions. Previous research in this direction shows that, by assuming that each consumer’s valuations on different items are drawn independently, approximation algorithms with better factors exist in some cases [16, 19, 20].

The second direction, which complements the first approach when input distributions cannot be assumed, is to add assumptions that allow us to exploit more problem structures. One such assumption is the price ladder constraint, as studied in [46, 1, 47, 48, 15]. Moreover, one might also consider the case where the size of each consideration set is small, e.g. [5, 15], or assume that consideration sets are *paths* in a graph, e.g., [5, 26, 34, 25, 30].

The observation that consumers make decisions based on attributes has been used in other areas outside computer science. For example, most pricing models are captured by the *two-stage consider-*

then-choose model (e.g., [32, 44, 45, 33, 36, 40, 37, 43]) in marketing research: Each consumer first screens out some undesirable items (*screening process*) and is left with the consideration set which is used to make a final decision. Pricing problems such as UUDP-MIN are the case where consideration sets are arbitrary (as defined in, e.g. [52, 38]) while the final decision is simplified to, e.g., buying the cheapest item.

The idea of using the consideration sets defined from attributes is called *attribute-based screening process* [33] in marketing research where it is shown to be a rational choice for trading off between accuracy and cognitive effort [9, 10, 11, 53]. Our model is equivalent to the attribute-based screening process with *conjunctive screening rules* (see, e.g., [33, 43]). This type of rules was justified by many studies that it is what consumers typically use when making decisions (see, e.g., [9, 33, 37]).

2 Algorithm for d -UUDP-MIN

We present the algorithm for d -UUDP-MIN now. Let \mathcal{C} and \mathcal{I} be the set of points in \mathbb{R}^d , where every consumer $\mathbf{C} \in \mathcal{C}$ has budget $B_{\mathbf{C}}$ and consideration set $S_{\mathbf{C}}$ which is specified by coordinates of the input point. For any subset $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{I}' \subseteq \mathcal{I}$, let $\mathcal{P}(\mathcal{C}', \mathcal{I}')$ be the d -UUDP-MIN problem with input \mathcal{C}' and \mathcal{I}' . Moreover, for any \mathcal{C}' and \mathcal{I}' , we use $\text{OPT}(\mathcal{C}', \mathcal{I}')$ to express the optimal revenue of the instance $(\mathcal{C}', \mathcal{I}')$. At a high level, our algorithm proceeds in an inductive manner and obtains a solution of d -UUDP-MIN problem by invoking the algorithms for $(d-1)$ -UUDP-MIN and 1-UUDP-MIN as a subroutine. Our result is summarized in the following theorem.

Theorem 2.1. *For any $\epsilon \in (0, 1]$, if there is an $\tilde{O}_d(n^{1-\epsilon})$ -approximation algorithm for $(d-1)$ -UUDP-MIN then there is an $\tilde{O}_d(n^{1-\epsilon/4})$ -approximation algorithm for d -UUDP-MIN as well.*

Theorem 1.1 then follows from the fact that 1-UUDP-MIN can be solved optimally in polynomial time (see Appendix A.5). As we noted earlier, it can be improved slightly since 2-UUDP-MIN admits QPTAS (see Appendix B).

2.1 Consideration-preserving Decomposition

Our algorithm partitions the input instance into many subinstances and tries to collect the profit from some of them. The notion of consideration-preserving decomposition, defined below, allows us to do so without losing revenue.

Definition 2.2. We call a collection $\{(\mathcal{C}'_1, \mathcal{I}'_1), \dots, (\mathcal{C}'_k, \mathcal{I}'_k)\}$ a *consideration-preserving decomposition* of the problem $(\mathcal{C}, \mathcal{I})$ if and only if for any $\mathbf{C} \in \mathcal{C}$ and $\mathbf{I} \in S_{\mathbf{C}}$, there exists (not necessarily unique) i such that $\mathbf{C} \in \mathcal{C}'_i$ and $\mathbf{I} \in \mathcal{I}'_i$.

By definition, for any consumer \mathbf{C} and item \mathbf{I} the fact that consumer \mathbf{C} considers item \mathbf{I} is preserved by at least one instance $(\mathcal{C}'_i, \mathcal{I}'_i)$. The following lemma says that this decomposition preserves the total revenue.

Lemma 2.3. *For any consideration-preserving decomposition $\{(\mathcal{C}'_1, \mathcal{I}'_1), \dots, (\mathcal{C}'_k, \mathcal{I}'_k)\}$ of $(\mathcal{C}, \mathcal{I})$, it holds that*

$$\sum_{i=1}^k \text{OPT}(\mathcal{C}'_i, \mathcal{I}'_i) \geq \text{OPT}(\mathcal{C}, \mathcal{I}).$$

Moreover, any price function for $\mathcal{P}(\mathcal{C}'_i, \mathcal{I}'_i)$ can be extended to a price function for the original problem $\mathcal{P}(\mathcal{C}, \mathcal{I})$ that gives revenue at least $\text{OPT}(\mathcal{C}'_i, \mathcal{I}'_i)$.

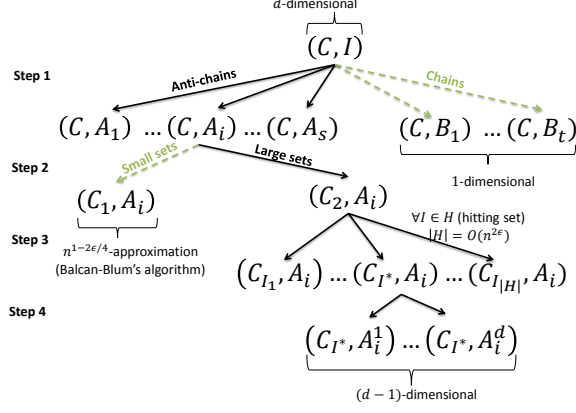


Figure 2: Decomposition overview

This is simply by applying the optimal price function of one problem to the other (see Appendix A.1 for the full proof). In the rest of our discussion, we mainly use two different types of consideration-preservation decomposition, as explained in the following observation.

Observation 2.4. *Given an input instance (C', \mathcal{I}') , let $C' = \bigcup_{i=1}^k C'_i$. Then $\{(C'_1, \mathcal{I}'), \dots, (C'_k, \mathcal{I}')$ is a consideration-preserving decomposition of (C', \mathcal{I}') . Similarly, if $\mathcal{I}' = \bigcup_{i=1}^k \mathcal{I}'_i$, then we have that $\{(C', \mathcal{I}'_1), \dots, (C', \mathcal{I}'_k)\}$ is a consideration-preserving decomposition of (C', \mathcal{I}') .*

2.2 Algorithm

At a high level, the algorithm proceeds in four steps where each step involves consideration-preserving decomposition (see Fig. 2 for an overview). In Step 1, we partition \mathcal{I} into different subsets where every subset satisfies certain properties, i.e. the elements in each subset either form a chain or an antichain. The problem on those subsets in which elements form a chain can be solved easily, and we deal with the antichains in later steps. In Step 2, we partition consumers in \mathcal{C} into two types, those with large and small consideration sets. We use the algorithm of [5, 15] to deal with consumers with small consideration sets and handle the rest consumers in later steps. In Step 3, we find a subset of items, i.e. a “hitting set”, and partition consumers further into several sets. Each set of consumers has the following property: There is some item desired by all consumers in the set. Using this property, we show in Step 4 that the problem can be further partitioned into a few problems where each of them can be viewed as a $(d-1)$ -UUDP-MIN problem. (We call this a “consideration-preserving embedding”.)

Step 1: Partitioning items into chains and antichains Let (C, \mathcal{I}) be an input of d -UUDP-MIN. First we define a partially ordered set (\mathcal{I}, \leq) on the item set as follows. We say that $\mathbf{I}_1 \leq \mathbf{I}_2$ if and only if \mathbf{I}_1 has a lower quality than \mathbf{I}_2 in every attribute, i.e. $\mathbf{I}_1[d'] \leq \mathbf{I}_2[d']$ for all $d' \in [d]$. We say that a subset $\mathcal{I}' \subseteq \mathcal{I}$ is a chain if \mathcal{I}' can be written as $\mathcal{I}' = \{\mathbf{I}_1, \dots, \mathbf{I}_z\}$ such that $\mathbf{I}_j \leq \mathbf{I}_{j+1}$ for all $j \in [z-1]$. We say that $\mathcal{I}' \subseteq \mathcal{I}$ is an antichain if and only if for any pair of items $\mathbf{I}, \mathbf{I}' \in \mathcal{I}'$, neither $\mathbf{I} \leq \mathbf{I}'$ nor $\mathbf{I}' \leq \mathbf{I}$.

Lemma 2.5. *For any $\epsilon > 0$ and any $s = n^{\epsilon/4}, t = n^{1-\epsilon/4}$, we can partition \mathcal{I} into A_1, \dots, A_s and B_1, \dots, B_t in polynomial-time. Moreover, each A_i is an antichain and each B_j is a chain.*

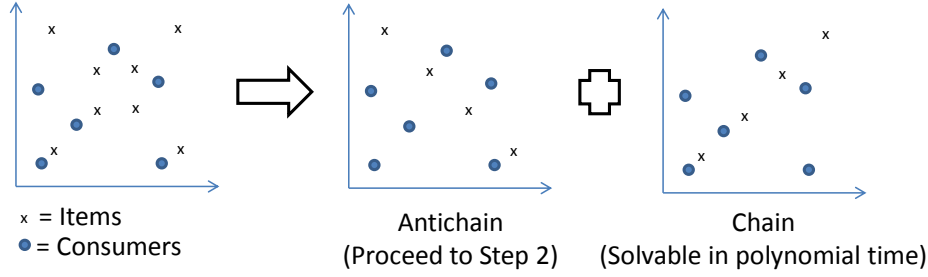


Figure 3: Example of Step 1

idea. (See Section A.2 for detailed definitions and proofs.) By Dilworth's theorem [24, 29], the minimum chain decomposition equals to the maximum antichain size. We will use the fact that both minimum chain decomposition and maximum-size antichain can be computed in polynomial time as follows: As long as the maximum-size antichain is bigger than $n^{\epsilon/4}$, we repeatedly extract such an antichain out of the input; otherwise, we would have the decomposition into at most $n^{\epsilon/4}$ chains, so we stop. \square

By Observation 2.4, the collection

$$\{(\mathcal{C}, A_1), \dots, (\mathcal{C}, A_s), (\mathcal{C}, B_1), \dots, (\mathcal{C}, B_t)\}$$

is a consideration-preserving decomposition of $(\mathcal{C}, \mathcal{I})$. It follows by Lemma 2.3 that

$$\sum_{i=1}^s \text{OPT}(\mathcal{C}, A_i) + \sum_{j=1}^t \text{OPT}(\mathcal{C}, B_j) \geq \text{OPT}(\mathcal{C}, \mathcal{I})$$

Further, observe that if there exists j such that $\text{OPT}(\mathcal{C}, B_j) \geq \text{OPT}(\mathcal{C}, \mathcal{I})/(2n^{1-\epsilon/4})$, then we would be done: the d -UUDP-MIN problem $\mathcal{P}(\mathcal{C}, B_j)$ can be seen as a 1-UUDP-MIN problem (since B_j is a chain) and hence can be solved optimally! (See Lemma 2.11 for detailed analysis) Otherwise $\text{OPT}(\mathcal{C}, B_j) \leq \text{OPT}(\mathcal{C}, \mathcal{I})/(2n^{1-\epsilon/4})$ for every j . Therefore

$$\sum_{j=1}^t \text{OPT}(\mathcal{C}, B_j) \leq n^{1-\epsilon/4} \cdot \text{OPT}(\mathcal{C}, \mathcal{I})/(2n^{1-\epsilon/4}) < \text{OPT}(\mathcal{C}, \mathcal{I})/2$$

If this is not the case then we know that there must be an antichain A_i such that

$$\text{OPT}(\mathcal{C}, A_i) \geq \text{OPT}(\mathcal{C}, \mathcal{I})/2n^{\epsilon/4}.$$

Step 2: Dealing with small consideration sets For simplicity, let us assume that we know i such that $\text{OPT}(\mathcal{C}, A_i) \geq \text{OPT}(\mathcal{C}, \mathcal{I})/(2n^{\epsilon/4})$. Now we focus on collecting revenue from the subproblem $\mathcal{P}(\mathcal{C}, A_i)$. Let $\mathcal{C}_1 \subseteq \mathcal{C}$ be the set of consumers who are interested in at most $n^{1-2\epsilon/4}$ items in A_i , i.e. $\mathcal{C}_1 = \{\mathbf{C} \in \mathcal{C} : |S_{\mathbf{C}} \cap A_i| \leq n^{1-2\epsilon/4}\}$, and define $\mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1$. Since $\{(\mathcal{C}_1, A_i), (\mathcal{C}_2, A_i)\}$ is a consideration-preserving decomposition of (\mathcal{C}, A_i) , we have

$$\text{OPT}(\mathcal{C}_1, A_i) + \text{OPT}(\mathcal{C}_2, A_i) \geq \text{OPT}(\mathcal{C}, A_i) \geq \frac{\text{OPT}(\mathcal{C}, \mathcal{I})}{2n^{\epsilon/4}}.$$

Now we need an algorithm of [5, 15]. Balcan and Blum give an approximation algorithm for SMP whose approximation guarantee depends on the sizes of consideration sets. Briest and Krysta, by using a slight modification of this algorithm, give an approximation algorithm with the same guarantee for UDP-MIN. Their result, stated in terms of UUDP-MIN, is summarized in the following theorem. (For completeness, we provide the proof in Appendix A.3.)

Theorem 2.6. [5, 15] *Given a UUDP-MIN instance $(\mathcal{C}, \mathcal{I}, \{S_{\mathbf{C}}\}_{\mathbf{C} \in \mathcal{C}})$, there is a deterministic $O(k)$ -approximation algorithm of UUDP-MIN, where $k := \max_{\mathbf{C} \in \mathcal{C}} |S_{\mathbf{C}}|$.*

We remark that we extend this technique to deal with any pricing problem with subadditive revenue in the full version of this paper.

If $\text{OPT}(\mathcal{C}_1, A_i) \geq \text{OPT}(\mathcal{C}, \mathcal{I})/(4n^{\epsilon/4})$, then we could invoke the algorithm in Theorem 2.6 on (\mathcal{C}_1, A_i) to get a solution with approximation ratio

$$O\left(\max_{\mathbf{C} \in \mathcal{C}_1} |S_{\mathbf{C}} \cap A_i|\right) = O(n^{1-2\epsilon/4}).$$

This yields a solution that gives a desired revenue of

$$\Omega\left(\text{OPT}(\mathcal{C}_1, A_i)/n^{1-2\epsilon/4}\right) = \Omega\left(\text{OPT}(\mathcal{C}, \mathcal{I})/n^{1-\epsilon/4}\right).$$

Otherwise we have $\text{OPT}(\mathcal{C}_1, A_i) < \text{OPT}(\mathcal{C}, \mathcal{I})/4n^{\epsilon/4}$. Then

$$\text{OPT}(\mathcal{C}_2, A_i) = \Omega\left(\text{OPT}(\mathcal{C}, \mathcal{I})/n^{\epsilon/4}\right).$$

We will deal with this case in the next steps.

Step 3: Partitioning consumers using a small hitting set First, we apply the epsilon net theorem [21, 39] to derive the following lemma.

Lemma 2.7. *We can find a set $H \subseteq A_i$ of size $\tilde{O}(n^{2\epsilon})$ in randomized polynomial time such that for any $\mathbf{C} \in \mathcal{C}_2$, there exists $\mathbf{I} \in H$ such that $\mathbf{I} \geq \mathbf{C}$.*

Proof. The instance (\mathcal{C}_2, A_i) defines a set system $\{S_{\mathbf{C}}\}_{\mathbf{C} \in \mathcal{C}_2}$ over A_i , where $S_{\mathbf{C}} = \{\mathbf{I} \in A_i \mid \mathbf{I} \geq \mathbf{C}\}$. We note that each set $S_{\mathbf{C}}$ has *descriptive complexity* at most d , i.e. set $S_{\mathbf{C}}$ can be described by d linear inequalities of the form $S_{\mathbf{C}} = \bigcap_{d'=1}^d \{\mathbf{I} \in \mathcal{I} : \mathbf{I}[d'] \geq \mathbf{C}[d']\}$. In this case, this set system has VC dimension $O(d)$, c.f. [51]. More specifically, it is well known (e.g., [4]) that any collection of d -dimensional axis-parallel boxes has VC dimension $O(d)$. We will not formally define VC-dimension here. The following theorem is all we need.

Theorem 2.8. ([21, 39]; *Epsilon net theorem*) *Let \mathcal{X} be a set system of VC-dimension at most d' over N . Then for any $\delta \in (0, 1)$, we can find a set $H \subseteq N$ with $|H| = O(\frac{d'}{\delta} \log \frac{d'}{\delta})$ in randomized polynomial time such that, for all $X_i \in \mathcal{X}$ with $|X_i| \geq \delta |N|$, it holds that $H \cap X_i \neq \emptyset$.*

Using the theorem with $\delta = n^{-2\epsilon/4}$, we can find a set $H \subseteq A_i$ of size at most $\tilde{O}(n^{2\epsilon/4})$, and since we have $|S_{\mathbf{C}} \cap A_i| \geq \delta n$ for all $\mathbf{C} \in \mathcal{C}_2$, we are guaranteed that $H \cap S_{\mathbf{C}} \neq \emptyset$ for all $\mathbf{C} \in \mathcal{C}_2$. \square

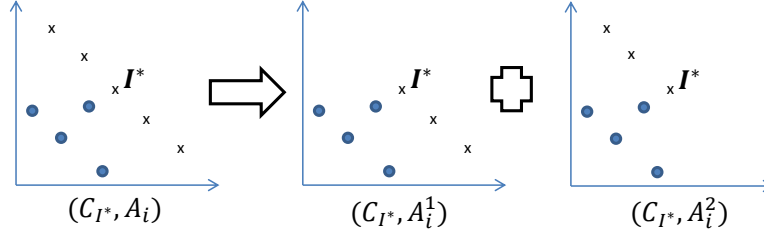


Figure 4: Example of Step 4

We call H a *hitting set* of \mathcal{C}_2 since H intersects $S_{\mathbf{C}}$ for all $\mathbf{C} \in \mathcal{C}_2$. We use H to decompose (\mathcal{C}_2, A_i) into a small number of subproblems and show in Step 4 that each of these problems can be viewed as a $(d-1)$ -UUDP-MIN problem.

For each $\mathbf{I} \in H$, let $\mathcal{C}_{\mathbf{I}} = \{\mathbf{C} \in \mathcal{C}_2 \mid \mathbf{I} \in S_{\mathbf{C}}\}$, i.e., $\mathcal{C}_{\mathbf{I}}$ consists of all consumers in \mathcal{C}_2 that consider item \mathbf{I} . Observe that $\bigcup_{\mathbf{I} \in H} \mathcal{C}_{\mathbf{I}} = \mathcal{C}_2$, and therefore by Lemma 2.3, we have

$$\sum_{\mathbf{I} \in H} \text{OPT}(\mathcal{C}_{\mathbf{I}}, A_i) \geq \text{OPT}(\mathcal{C}_2, A_i) \geq \Omega\left(\text{OPT}(\mathcal{C}, \mathcal{I})/n^{\epsilon/4}\right).$$

Since $|H| = O(n^{2\epsilon/4})$, there exists $\mathbf{I}^* \in H$ such that

$$\begin{aligned} \text{OPT}(\mathcal{C}_{\mathbf{I}^*}, A_i) &= \tilde{\Omega}\left(\text{OPT}(\mathcal{C}, \mathcal{I}) \cdot n^{-\epsilon/4}/|H|\right) \\ &= \tilde{\Omega}\left(\text{OPT}(\mathcal{C}, \mathcal{I})/n^{3\epsilon/4}\right). \end{aligned}$$

Now we, again, assume that we know \mathbf{I}^* and turn our focus to the subproblem $\mathcal{P}(\mathcal{C}_{\mathbf{I}^*}, A_i)$.

Step 4: Reducing the dimension We have now reached the most crucial step. We will (crucially) rely on the fact that all consumers in $\mathcal{C}_{\mathbf{I}^*}$ consider item \mathbf{I}^* , and that A_i is an antichain.

For each $j \leq d$, define A_i^j as the set of items in A_i that are at least as good as \mathbf{I}^* in the j -th coordinate, i.e., $A_i^j = \{\mathbf{I} \in A_i \mid \mathbf{I}[j] \geq \mathbf{I}^*[j]\}$. See Fig. 4 for an example in the case of 2-UUDP-MIN.

Lemma 2.9. $A_i = \bigcup_{j=1}^d A_i^j$.

This lemma holds simply because A_i is an antichain (in any antichain, no item can completely dominate the others, so at least one coordinate of any $\mathbf{I} \in \mathcal{I}_{\mathbf{I}^*}$ has to be at least as good as \mathbf{I}^* ; see detailed proof in Appendix A.4). Then $\{(\mathcal{C}_{\mathbf{I}^*}, A_i^1), \dots, (\mathcal{C}_{\mathbf{I}^*}, A_i^d)\}$ is a consideration-preserving decomposition of $(\mathcal{C}_{\mathbf{I}^*}, A_i)$ and thus there exists j such that

$$\begin{aligned} \text{OPT}(\mathcal{C}_{\mathbf{I}^*}, A_i^j) &\geq \text{OPT}(\mathcal{C}_{\mathbf{I}^*}, A_i)/d \\ &= \tilde{\Omega}_d(\text{OPT}(\mathcal{C}, \mathcal{I})/n^{3\epsilon/4}). \end{aligned}$$

Observe that, for all $\mathbf{C} \in \mathcal{C}_{\mathbf{I}^*}$ and $\mathbf{I} \in A_i^j$, $\mathbf{C}[j] \leq \mathbf{I}^*[j] \leq \mathbf{I}[j]$. This implies that we can ignore the j -th coordinate when we solve $\mathcal{P}(\mathcal{C}_{\mathbf{I}^*}, A_i^j)$. (In particular, for any $\mathbf{C} \in \mathcal{C}_{\mathbf{I}^*}$, the consideration set

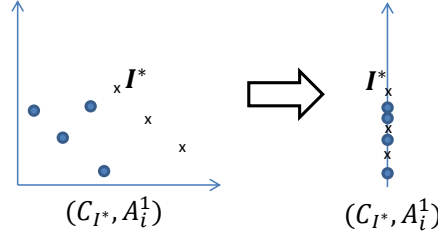


Figure 5: Example of Step 4 when we view the instance \mathcal{C}_{I^*}, A_i^j as a $(d-1)$ -UUDP-MIN instance. Observe that the consideration sets in both pictures are the same.

$S_{\mathbf{C}} = \{\mathbf{I} \geq \mathbf{C} \mid \mathbf{I} \in A_i^j\}$ remains the same even when we drop the j -th coordinate of all points.) In other words, the problem can be viewed as a $(d-1)$ -UUDP-MIN problem (see Fig. 5 for an idea). We defer the formal statement and proof of this claim to Section 2.3.

Finally, we can invoke the $\tilde{O}_d(n^{1-\epsilon})$ -approximation algorithm for $(d-1)$ -UUDP-MIN to collect the revenue of

$$\tilde{\Omega}_d \left(\text{OPT}(\mathcal{C}, \mathcal{I}) n^{-3\epsilon/4} / n^{1-\epsilon} \right) = \tilde{\Omega}_d \left(\text{OPT}(\mathcal{C}, \mathcal{I}) / n^{1-\epsilon/4} \right).$$

Therefore we obtain an approximation ratio of $\tilde{O}_d(n^{1-\epsilon/4})$ in all cases. Algorithm 1 summarizes our algorithm for solving d -UUDP-MIN.

2.3 Consideration-preserving Embedding

To formally discuss the reduction of dimensions, we introduce the notion of consideration-preserving embedding. For any d , let $(\mathcal{C}, \mathcal{I})$ be any instance of d -UUDP-MIN. For any d' , consider one-to-one functions f and g that map points in \mathbb{R}^d to the ones in $\mathbb{R}^{d'}$. We say that (f, g) is a *consideration-preserving embedding* if, for any item $\mathbf{I} \in \mathcal{I}$ and consumer $\mathbf{C} \in \mathcal{C}$, we have that $\mathbf{I} \geq \mathbf{C}$ if and only if $g(\mathbf{I}) \geq f(\mathbf{C})$. That is, the fact that consumer \mathbf{C} is considering or not considering item \mathbf{I} must be preserved in $f(\mathbf{C})$ and $g(\mathbf{I})$.

Given a consideration-preserving embedding (f, g) , we can naturally define a d' -UUDP-MIN problem $\mathcal{P}(f(\mathcal{C}), g(\mathcal{I}))$ where $f(\mathcal{C}) = \{f(\mathbf{C}) \mid \mathbf{C} \in \mathcal{C}\}$, $g(\mathcal{I}) = \{g(\mathbf{I}) \mid \mathbf{I} \in \mathcal{I}\}$ and the budget $B_{f(\mathbf{C})}$ is $B_{\mathbf{C}}$ for any $\mathbf{C} \in \mathcal{C}$.

Observe that, although $(\mathcal{C}, \mathcal{I})$ and $(f(\mathcal{C}), g(\mathcal{I}))$ correspond to points on different spaces, they represent the same pricing problem (i.e., the consumers' consideration sets and budgets are exactly the same). Thus, we sometimes say that $(\mathcal{C}, \mathcal{I})$ and $(f(\mathcal{C}), g(\mathcal{I}))$ are *equivalent*. The following observation follows trivially.

Observation 2.10. *For any instance $(\mathcal{C}, \mathcal{I})$, let (f, g) be a consideration-preserving embedding of $(\mathcal{C}, \mathcal{I})$ into $\mathbb{R}^{d'}$. Then we have that*

$$\text{OPT}(\mathcal{C}, \mathcal{I}) = \text{OPT}(f(\mathcal{C}), g(\mathcal{I}))$$

Moreover, if f and g are polynomial-time computable then a solution for $\mathcal{P}(f(\mathcal{C}), g(\mathcal{I}))$ can be efficiently transformed into one for $\mathcal{P}(\mathcal{C}, \mathcal{I})$ that gives the same revenue.

Algorithm 1 UUDP-MIN-APPROX(d)

```
1: if  $d = 1$  then
2:   Solve the problem  $\mathcal{P}(\mathcal{C}, \mathcal{I})$  optimally using an algorithm for 1-UUDP-MIN (cf. Appendix A.5)
3: else
4:   Partition  $\mathcal{I}$  into antichains  $A_1, \dots, A_s$  and chains  $B_1, \dots, B_t$  where  $s \leq n^{\epsilon/4}$  and  $t \leq n^{1-\epsilon/4}$ 
      as in Step 1.
5:   We claim that the problems  $\mathcal{P}(\mathcal{C}, B_1), \dots, \mathcal{P}(\mathcal{C}, B_t)$  are equivalent to 1-UUDP-MIN problems
      (cf. Section A.2). Solve them optimally using an algorithm for 1-UUDP-MIN (cf. Ap-
      pendix A.5).
6:   for  $i = 1, \dots, s$  do
7:     Partition  $\mathcal{C}$  into  $\mathcal{C}_1$  and  $\mathcal{C}_2$  as in Step 2. Find an  $O(\max_{\mathbf{C} \in \mathcal{C}_1} |S_{\mathbf{C}} \cap A_i|) = O(n^{1-2\epsilon/4})$  ap-
      proximate solution of problem  $\mathcal{P}(\mathcal{C}_1, A_i)$ .
8:     Find a hitting set  $H$  of  $(\mathcal{C}_2, A_i)$  as in Step 3
9:     for each  $\mathbf{I} \in H$  do
10:      Define  $\mathcal{C}_{\mathbf{I}}$  as in Step 3
11:      Define  $A_i^1, \dots, A_i^d$  as in Step 4
12:      Solve problem  $\mathcal{P}(\mathcal{C}_{\mathbf{I}}, A_i^1), \dots, \mathcal{P}(\mathcal{C}_{\mathbf{I}}, A_i^d)$  using an  $O(n^{1-\epsilon})$ -approximation algorithm for
       $(d-1)$ -UUDP-MIN
13:    end for
14:  end for
15: end if
16: return the solution with highest revenue among the solutions of all solved problems
```

The transformation in the above lemma is trivial: For any price function p for $(f(\mathcal{C}), g(\mathcal{I}))$, we simply price item $\mathbf{I} \in \mathcal{I}$ to $p(g(\mathbf{I}))$. Observe that we will receive the same revenue from both problems using this pricing strategy.

In Step 1, we claimed that when the items form a chain, our instance would be equivalent to 1-UUDP-MIN. Now we prove this fact formally below.

Lemma 2.11. *Let $(\mathcal{C}, \mathcal{I})$ be a d -UUDP-MIN instance where (\mathcal{I}, \leq) is a chain. Then $(\mathcal{C}, \mathcal{I})$ is equivalent to a 1-UUDP-MIN instance. Moreover, the corresponding consideration-preserving embedding (f, g) can be computed in polynomial time.*

Proof. Order items in \mathcal{I} by $\mathbf{I}_1 \leq \mathbf{I}_2 \leq \dots$. Now map each item into a one-dimensional point: $g(\mathbf{I}_i) = (i)$. Moreover, map each consumer according to $f(\mathbf{C}) = g(\mathbf{I}_i)$, where i is the minimum number such that $\mathbf{I}_i \geq \mathbf{C}$. Observe that (f, g) is a consideration-preserving embedding since $S_{\mathbf{C}} = \{\mathbf{I}_i, \mathbf{I}_{i+1}, \dots\}$ while $S_{f(\mathbf{C})} = \{g(\mathbf{I}_i), g(\mathbf{I}_{i+1}), \dots\}$ for any $\mathbf{C} \in \mathcal{C}$. (Note that this embedding might create redundancy since it is possible that $f(\mathbf{C}) = f(\mathbf{C}')$ for some $\mathbf{C} \neq \mathbf{C}'$. This can be fixed easily by slightly perturbing the points.) \square

In Step 4, we also claimed the dimension reduction of sub-instances $(\mathcal{C}_{\mathbf{I}^*}, A_i^j)$, and we now prove the claim formally. Recall that the item $\mathbf{I}^* \in A_i^j$ has the property that $\mathbf{I}^* \geq \mathbf{C}$ for all $\mathbf{C} \in \mathcal{C}_{\mathbf{I}^*}$ and $\mathbf{I}^*[j] \leq \mathbf{I}[j]$ for all $\mathbf{I} \in A_i^j$.

Lemma 2.12. *The instance $(\mathcal{C}_{\mathbf{I}^*}, A_i^j)$ is equivalent to a $(d-1)$ -UUDP-MIN instance. Moreover, the corresponding consideration-preserving embedding (f, g) can be computed in polynomial time.*

Proof. Consider “ignoring” the j -th coordinate as follows. For any $\mathbf{C} \in \mathcal{C}_{\mathbf{I}^*}$ and $\mathbf{I} \in A_i^j$, let

$$f(\mathbf{C}) = (\mathbf{C}[1], \mathbf{C}[2], \dots, \mathbf{C}[j-1], \mathbf{C}[j+1], \dots, \mathbf{C}[d])$$

and

$$g(\mathbf{I}) = (\mathbf{I}[1], \mathbf{I}[2], \dots, \mathbf{I}[j-1], \mathbf{I}[j+1], \dots, \mathbf{I}[d]).$$

Observe that for any $\mathbf{C} \in \mathcal{C}_{\mathbf{I}^*}$ and $\mathbf{I} \in A_i^j$, $\mathbf{I} \geq \mathbf{C}$ trivially implies that $g(\mathbf{I}) \geq f(\mathbf{C})$. Conversely, if $g(\mathbf{I}) \geq f(\mathbf{C})$ then $\mathbf{I} \geq \mathbf{C}$ since $\mathbf{I}[j] \geq \mathbf{I}^*[j] \geq \mathbf{C}[j]$. Thus, (f, g) is a consideration-preserving embedding. \square

3 Hardness

We provide hardness results in both scenarios when the number of attributes d is small and when d is large. We sketch our results here. More details can be found in Appendix C.

Few attributes First we discuss the NP-hardness of 3-UUDP-MIN and APX-hardness of 4-UUDP-MIN. These hardness results hold even when the consumer budgets are either 1 or 2. We perform a reduction from Vertex Cover [31, 3], essentially using the same ideas as in [35], except for the fact that we use Schnyder’s result [49, 50] to “embed” the instance into posets of low order dimensions.

First, let us recall the reduction in [35]. We start from a graph $G = (V, E)$, which is an input instance of Vertex Cover. We create two types of consumers: (i) poor consumer \mathbf{C}_e for each edge e with budget 1 and (ii) rich consumer \mathbf{C}_v for each vertex v with budget 2. The items are $\mathcal{I} = \{\mathbf{I}_v : v \in V\}$. Each poor consumer \mathbf{C}_e has a consideration set containing two items \mathbf{I}_u and \mathbf{I}_v where $e = (u, v)$ and each rich consumer \mathbf{C}_v considers only one item \mathbf{I}_v . Using the analysis essentially the same as [35], one can show that the problem is NP-hard if we start from Vertex Cover on planar graphs and APX-hard if we start from Vertex Cover on cubic graphs.

Therefore, it only remains to map consumers and items to points in $\mathbb{R}_{\geq 0}^d$ (where $d = 3, 4$) such that for each consumer \mathbf{C} , the set of items that pass her criteria (i.e., $\{\mathbf{I} \in \mathcal{I} \mid \mathbf{I}[i] \geq \mathbf{C}[i] \text{ for all } 1 \leq i \leq d\}$) is exactly her consideration set. The main idea is to first embed the problem into an *adjacency poset* of the input graph. Then, we invoke Schnyder’s theorem [49, 50] to again embed this poset into a Euclidean space.

An adjacency poset of a graph can be constructed as follows. First we construct a 2-layer poset with minimal elements in the first layer and maximal elements in the second layer. For each edge $e \in E$, we have a minimal element in the poset corresponding to e (for convenience, we also denote the poset element by e). For each vertex $v \in V$, we have a maximal poset element corresponding to v . There is a relation $e \preceq v$ if and only if vertex v is an endpoint of e .

The last task is to “embed” poset elements into points in the Euclidean space in such a way that, for any poset elements e_1 and e_2 , $e_1 \preceq e_2$ if and only if $q_{e_1}[i] \geq q_{e_2}[i]$ for all i where q_{e_1} and q_{e_2} are points that e_1 and e_2 are mapped to, respectively. If we can do this, we would be done, simply by defining the coordinates of each consumer \mathbf{C}_e to be q_e , and the coordinates of each consumer \mathbf{C}_v to be q_v . Similarly, we define the coordinates of each item \mathbf{I}_v as q_v . In order to obtain such an embedding, we use part of Schnyder’s theorem [49] which states that any planar graph has an adjacency poset of dimension three, and any 4-colorable graph (including cubic graphs) has an adjacency poset of dimension four. Moreover, embedding these graphs into Euclidean spaces can be done in polynomial time [50].

Finally we note that 2-SMP is strongly NP-hard and 4-SMP is APX-hard. The proof follows from the fact that these problems generalize Highway pricing and graph vertex pricing on bipartite graphs, respectively, and can be found in the full version.

Many attributes We establish a connection between the UUDP-MIN with bounded-size consideration sets and our problem. This connection immediately implies hardness results for d -UUDP-MIN when d is at least poly-logarithmic in n . Our main result in this section is the following:

Theorem 3.1. *(Informal) Let $A = (\mathcal{C}, \mathcal{I}, \{S_C\}_{C \in \mathcal{C}})$ be an instance of UUDP-MIN where $B = \max_{C \in \mathcal{C}} |S_C|$. We can (with high probability of success) create an instance $A' = (\mathcal{C}', \mathcal{I}')$ of d -UUDP-MIN, where $d = O(B^2 \log n)$, that is “equivalent” to A .*

In other words, the above theorem shows that any UUDP-MIN instance with consideration sets of size bounded by B , can be realized by a d -UUDP-MIN instance for $d = O(B^2 \log n)$. Combining this with the result in [13], we have a hardness of $\Omega(d^\epsilon)$ for some $\epsilon > 0$, assuming the hardness of the balanced bipartite independent set problem in constant degree graphs or refuting random 3CNF formulas.

We remark that our reduction here in fact works independently of the decision model, so this result works for SMP and UDP-Util as well.

4 Open Problems

We formulated the multi-attribute pricing problem in this paper and considered the approximation and hardness results for several variations. Several interesting problems are open. Of course, the most important problem is whether we can obtain better approximation factors for interesting models of the d -attribute pricing problem (e.g., d -UUDP-MIN, d -SMP, d -attribute UDP-Util, etc.). Among these, the two simplest models, d -UUDP-MIN and d -SMP, are most likely to admit good approximation ratios.

We tend to believe that there is an $f(d)$ -approximation algorithm for d -UUDP-MIN and d -SMP where $f(d)$ is a function that depends on d only. However, it seems to be a very challenging task to obtain approximation ratio like $\log^{O(d)} n$ or $O_d(\log^{1-\epsilon(d)} m)$, for some constant $\epsilon(d) > 0$ depending on d .

One promising direction in attacking the above problems is to improve Theorem 2.1, e.g., getting $O_d(\rho \cdot \text{polylog}(n))$ for d -UUDP-MIN using a ρ -approximation algorithm of $(d-1)$ -UUDP-MIN as a blackbox. A positive resolution to this problem would imply $(\log^{O(d)} n)$ -approximation algorithm for d -UUDP-MIN. We believe that, even resolving this problem would require some new insights on geometric and poset structures.

There are two special cases that can be thought of as barriers in dealing with standard versions of SMP and UUDP-MIN, and we believe that these two special cases serve as good starting points in attacking our problems. The first problem is the d -attribute version of the Maximum Expanding Subsequence (MES) problem which is the key problem to show the hardness of UUDP-MIN [14]. The second problem is the Unique Coverage problem [23] when the sets have constant VC-dimension. Another interesting problem is to obtain PTASs for 2-UUDP-MIN and 2-SMP (e.g., by extending the techniques in [34]). We refer to the full version for details of these and other open problems.

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Appendix

A Proof Omitted from Section 2

A.1 Proof of Lemma 2.3

Let p^* be the optimal price function for $\mathcal{P}(\mathcal{C}, \mathcal{I})$. For each $i = 1, \dots, k$, we define $p_i^* : \mathcal{I}'_i \rightarrow \mathbb{R}$ by

$$p_i^*(\mathbf{I}) = p^*(\mathbf{I}) \text{ if } \mathbf{I} \in \mathcal{I}'_i, \text{ and } p_i^*(\mathbf{I}) = \infty \text{ otherwise.}$$

Let r_i be the total revenue made by p_i^* in $\mathcal{P}(\mathcal{C}'_i, \mathcal{I}'_i)$. We argue below that

$$\sum_{i=1}^k r_i \geq \text{OPT}(\mathcal{C}, \mathcal{I}). \quad (2)$$

Let $\mathcal{C}^* \subseteq \mathcal{C}$ be the set of consumers who make a positive payment with respect to p^* . For each consumer $\mathbf{C} \in \mathcal{C}^*$, denote by $\varphi(\mathbf{C}) \in \mathcal{I}$ the item that consumer \mathbf{C} buys with respect to the price p^* . So we can write $\text{OPT}(\mathcal{C}, \mathcal{I})$ as

$$\text{OPT}(\mathcal{C}, \mathcal{I}) = \sum_{\mathbf{C} \in \mathcal{C}^*} p^*(\varphi(\mathbf{C})). \quad (3)$$

For each $i = 1, \dots, k$, let $\mathcal{C}_i^* \subseteq \mathcal{C}'_i$ be the set of consumers $\mathbf{C} \in \mathcal{C}'_i$ such that $\varphi(\mathbf{C}) \in \mathcal{I}'_i$. That is, \mathcal{C}_i^* is a set of consumers whose item she bought in $\text{OPT}(\mathcal{C}, \mathcal{I})$ is in \mathcal{I}'_i . Notice that

$$r_i \geq \sum_{\mathbf{C} \in \mathcal{C}_i^*} p^*(\varphi(\mathbf{C})). \quad (4)$$

Since $\{(\mathcal{C}'_i, \mathcal{I}'_i)\}_{i=1}^k$ is a consideration-preserving decomposition, we have that

$$\bigcup_{i=1}^k \mathcal{C}_i^* \supseteq \mathcal{C}^*, \quad (5)$$

since for any $\mathbf{C} \in \mathcal{C}^*$, we must have $\varphi(\mathbf{C}) \in \mathcal{I}_i$ for some i . By summing Eq.(4) over all $i = 1, \dots, k$, we have

$$\begin{aligned} \sum_{i=1}^k r_i &\geq \sum_{i=1}^k \sum_{\mathbf{C} \in \mathcal{C}_i^*} p^*(\varphi(\mathbf{C})) && \text{(by Eq.(4))} \\ &\geq \sum_{\mathbf{C} \in \mathcal{C}^*} p^*(\varphi(\mathbf{C})) && \text{(by Eq.(5))} \\ &= \text{OPT}(\mathcal{C}, \mathcal{I}) && \text{(by Eq.(3))} \end{aligned}$$

This proves Eq.(2) and thus the first claim.

Now suppose we have a price $p' : \mathcal{I}_i \rightarrow \mathbb{R}$ that collects revenue r' in $\mathcal{P}(\mathcal{C}'_i, \mathcal{I}'_i)$. We define a function $p : \mathcal{I} \rightarrow \mathbb{R}$ by $p(\mathbf{I}) = p'(\mathbf{I})$ for $\mathbf{I} \in \mathcal{I}'_i$ and $p(\mathbf{I}) = \infty$ otherwise. We can use p' to obtain a revenue of r' from $\mathcal{P}(\mathcal{C}, \mathcal{I})$. This proves the second claim.

A.2 Decomposing items into small number of chains and antichains

We will use the following theorem, first proved by Dilworth [24], and its polynomial computability follows from the equivalence between Dilworth's theorem and König's theorem [29].

Theorem A.1. *Let (S, \leq) be a partially ordered set, and Z be the maximum number of elements in any antichain of S . Then there is a polynomial-time algorithm that produces a partition of S into Z chains S_1, \dots, S_Z .*

We now use the theorem to prove Lemma 2.5.

of Lemma 2.5. Initially, let $i = 1$. In iteration i , we check if the size of maximum antichain in \mathcal{I} is at least $t = n^{1-\epsilon/4}$. If so, we find the maximum antichain A_i , update $\mathcal{I} = \mathcal{I} \setminus A_i$, and proceed to the next iteration; otherwise, we stop the iterations. Notice that the number of iterations is at most $s = n^{\epsilon/4}$, and when the iteration stops, the size of maximum-size antichain is at most $t \leq n^{1-\epsilon/4}$. We apply the above theorem to compute a decomposition of \mathcal{I} into t chains, denoted by B_1, \dots, B_t . This concludes the proof of Lemma 2.5. \square

A.3 Proof of Balcan-Blum Theorem for UUDP-MIN (cf. Theorem 2.6)

We first explain a randomized algorithm, and then we discuss how to derandomize it. This part is essentially the same as [5, 15]. First, we randomly construct a set $\mathcal{I}^* \subseteq \mathcal{I}'$ where each item \mathbf{I} is independently added to \mathcal{I}^* with probability $1/k$ (recall that $k = \max_{\mathbf{C} \in \mathcal{C}} |S_{\mathbf{C}}|$). Then let \mathcal{C}^* be a set of consumer \mathbf{C} such that $|S_{\mathbf{C}} \cap \mathcal{I}^*| = 1$ (i.e. consumers who care about exactly one item in \mathcal{I}^*). We show that the problem $\mathcal{P}(\mathcal{C}^*, \mathcal{I}^*)$ has expected revenue at least $\Omega(\text{OPT}(\mathcal{C}, \mathcal{I})/k)$.

Let p be the optimal price function for $(\mathcal{C}, \mathcal{I})$ and $\varphi : \mathcal{C} \rightarrow \mathcal{I} \cup \{\perp\}$ be a function that maps each consumer to the item she buys with respect to p (let $\varphi(\mathbf{C}) = \perp$ if consumer \mathbf{C} buys nothing and $p(\perp) = 0$). Therefore, we have that $\text{OPT}(\mathcal{C}, \mathcal{I}) = \sum_{\mathbf{C}} p(\varphi(\mathbf{C}))$. We denote by p^* the price function p restricted to \mathcal{I}^* . For each \mathbf{C} , if $\mathbf{C} \in \mathcal{C}^*$ and $\varphi(\mathbf{C}) \in \mathcal{I}^*$, the revenue created by p^* in $(\mathcal{C}^*, \mathcal{I}^*)$ would be at least $p(\varphi(\mathbf{C}))$. Therefore,

$$\mathbf{E} [\text{OPT}(\mathcal{C}^*, \mathcal{I}^*)] \geq \sum_{\mathbf{C} \in \mathcal{C}} \Pr[\varphi(\mathbf{C}) \in \mathcal{I}^* \text{ and } \mathbf{C} \in \mathcal{C}^*] \times p(\varphi(\mathbf{C})).$$

Notice that, for any $\mathbf{C} \in \mathcal{C}$ and $\mathbf{I} \in S_{\mathbf{C}}$,

$$\Pr[\mathbf{I} \in \mathcal{I}^* \text{ and } \mathbf{C} \in \mathcal{C}^*] \geq \frac{1}{k} \left(1 - \frac{1}{k}\right)^{k-1} \geq \frac{1}{ke},$$

which implies that $\mathbf{E} [\text{OPT}(\mathcal{C}^*, \mathcal{I}^*)] \geq \frac{1}{ke} \cdot \text{OPT}(\mathcal{C}, \mathcal{I})$.

Derandomization: First, note that we can assume that $k = O(\log m + \log n)$. Otherwise, we can use the result of [1, 35, 7] (see [7, Section 4] for the result in a general setting) to obtain $O(\log m + \log n)$ approximation algorithm for UUDP-MIN, which will also be $O(k)$ -approximation.

Now, assuming that $k = O(\log m + \log n)$, we follow the argument of Balcan and Blum [5]. In particular, we observe that we need only k -wise independence among the events of the form “ $\mathbf{I} \in \mathcal{I}^*$ and $\mathbf{C} \in \mathcal{C}^*$ ”, for any \mathbf{I} and \mathbf{C} , in order to get the above expectation result. In this case, we can use the tools from Even et al [27] to derandomize the above algorithm while blowing up the running time by a factor of $2^{O(k)} = \text{poly}(m, n)$. For more details, we refer the readers to [5].

A.4 Proof of Lemma 2.9

Recall that each A_i is an antichain, i.e., for any distinct $\mathbf{I}_1, \mathbf{I}_2 \in A_i$, there exists $1 \leq d_1, d_2 \leq d$ such that $\mathbf{I}_1[d_1] < \mathbf{I}_2[d_1]$ and $\mathbf{I}_1[d_2] > \mathbf{I}_2[d_2]$. In particular, if $\mathbf{I}_1 = \mathbf{I}^*$, then we have that for any $\mathbf{I} \in A_i$, there exists coordinate j such that $\mathbf{I}[j] \geq \mathbf{I}^*$. This means that $\mathbf{I} \in A_i^j$. The lemma follows.

A.5 Polynomial-Time Algorithm for 1-UUDP-MIN

We provide a polynomial-time algorithm for solving 1-UUDP-MIN. Let $\mathbf{I}_1, \dots, \mathbf{I}_n$ be a sequence of items ordered non-increasingly by their coordinates. We can assume without loss of generality that their coordinates are different (by slightly perturbing their values), and we say that consumer \mathbf{C} is at *level* j if her coordinate lies between \mathbf{I}_{j-1} and \mathbf{I}_j . Notice that, for any consumer \mathbf{C} at level j , we have $S_{\mathbf{C}} = \{\mathbf{I}_1, \dots, \mathbf{I}_j\}$.

Claim A.2. *Let p^* be an optimal price. Then we can assume that $p^*(\mathbf{I}_1) \geq p^*(\mathbf{I}_2) \geq \dots \geq p^*(\mathbf{I}_n)$.*

Proof. Suppose that $p^*(\mathbf{I}_i) < p^*(\mathbf{I}_j)$ for some $i < j$. Recall that $\mathbf{I}_i \geq \mathbf{I}_j$, so for each consumer \mathbf{C} such that $\mathbf{C} \leq \mathbf{I}_j$, we know that \mathbf{C} does not buy item \mathbf{I}_j with respect to this solution. Thus, we can reduce $p^*(\mathbf{I}_i)$ slightly, while maintaining the same revenue. \square

The claim will ensure that consumers at level j only buy item \mathbf{I}_j but not any other items in $\{\mathbf{I}_1, \dots, \mathbf{I}_{j-1}\}$, and this allows us to solve the problem by dynamic programming. For each $j = 1, \dots, n$, for each price $P \in \mathbb{R}$ we have a table entry $T[j, P]$ that keeps the maximum revenue achievable from consumers at levels $1, \dots, j$ and items $\{\mathbf{I}_1, \dots, \mathbf{I}_j\}$ where the price of \mathbf{I}_j is set to P . Notice that it is easy to compute the profit from consumers at level j if we know $p(\mathbf{I}_j) = P$. Denote such value by γ . Then we have that $T[j, P] = \gamma + \max_{P' \geq P} T[j-1, P']$. Finally, we note that there are at most $|\mathcal{C}|$ possibilities of prices P because one can assume without loss of generality that, for UUDP-MIN, the prices always belong to $\{B_{\mathbf{C}}\}_{\mathbf{C} \in \mathcal{C}}$.

B QPTAS for 2-UUDP-MIN

We note that we will write $O(\log m)$ instead of $O(\log n + \log m)$ since we assume that $n \leq m$ in this paper. (Otherwise, we already have approximation ratio of $O(\log m) = O(\log n)$.)

We explain the main idea first. The intuition can be realized by solving the following simple case: Assume for now that we have $\Theta(n^2)$ items, which form a set $\{(2i-1, 2j-1) : 1 \leq i, j \leq n\}$. In this case it is possible to have two different consumers at the same coordinate, i.e. $\mathbf{C} = \mathbf{C}'$, while there is exactly one item at each point $(2i-1, 2j-1)$. Assume further that each consumer has budget either 1 or 2. We show below how to solve this case in polynomial time.

Note that there is an optimal solution such that each item is priced either 1 or 2: otherwise we could increase the price by small amount to collect more revenue. Now, for any item point $(2i-1, 2j-1)$ and any price assignment p , define

$$r_p(i, j) := \min_{\substack{\mathbf{I}[1] \geq 2i-1, \mathbf{I}[2] \geq 2j-1 \\ \mathbf{I} \in \mathcal{I}}} \{p(\mathbf{I})\}$$

to be the minimum price among the items dominating $(2i-1, 2j-1)$. This quantity immediately tells us how much revenue we will get from consumers at point $(2i-2, 2j-2)$: each consumer will buy an item at price $r_p(i, j)$ if and only if she has budget at least $r_p(i, j)$.

By the definition of r_p , we know that for any fixed value j , $r_p(i, j)$ is non-decreasing in terms of i . In other words, for any pricing p and integer j , there exists a “threshold” $\gamma(p, j)$ such that $r_p(i', j) = 1$ for all $i' \leq \gamma(p, j)$ and $r_p(i', j) = 2$ for all $i' > \gamma(p, j)$. Additionally, for any j , $\gamma(p, j) \geq \gamma(p, j + 1)$. Using these observations, we are ready to define the dynamic programming table. The table entry $T[i, j]$ is defined to be the maximum revenue we can get among the price assignment p such that $r_p(i', j) = 1$ for all $i' \leq i$ and $r_p(i', j) = 2$ for all $i' > i$. The table T can be computed as follows.

$$T[i, j] = \max_{i' \leq i} \{T[i', j + 1] + m_1(i', j) + 2m_2(i', j)\} \quad (6)$$

where $m_1(i', j)$ is the number of consumers of the form $(2i'' - 2, 2j - 2)$ for $i'' \leq i'$ with budget 1 and $m_2(i', j)$ is the number of consumers of the form $(2i'' - 2, 2j - 2)$ for $i'' > i'$ with budget 2. Moreover, let $T[i, n + 1] = 0$ for all i . The optimal solution is then $\max_i T[i, 1]$.

The above discussion captures almost all the key ideas for solving the general 2-UUDP-MIN problem. To get a QPTAS in the general case, we extend these ideas in the following way.

- Consider a slight generalization when there is only one item in each column and row of grid cells (cf. Lemma B.1) while each budget is still 1 and 2. In this case, we cannot pick arbitrary value of i' when we compute $T[i, j]$ as in Eq.(6) since it might not correspond to any pricing. Through some additional observations, table T can be computed as follows: Let \mathbf{I}_j be the item whose y -coordinate is j . If $i = \mathbf{I}_j[1]$ then we can use Eq.(6); otherwise, $T[i, j] = T[i, j + 1] + m_1(i, j) + 2m_2(i, j)$. This algorithm runs in $O(n^3)$ time.
- When there are q different budgets, say B_1, B_2, \dots, B_q , we can solve the problem in $n^{O(q)}$ time. This is done by defining $T[i_1, \dots, i_{q-1}, j]$ to be the maximum revenue we can get among the price assignment p such that, for all $q' : 1 \leq q' \leq q$, $r_p(i', j) = B_{q'}$ for all $i_{q'-1} < i' \leq i_{q'}$ (where we let $i_0 = -1$ and $i_q = n$).
- Finally, we obtain a QPTAS by “discretizing” the prices so that there are not many choices of item prices (cf. Lemma B.2). This enables us to assume that the prices are in $\Gamma = \{0, (1 + \epsilon)^0, (1 + \epsilon)^1, \dots, (1 + \epsilon)^q\}$ where $q = O(\log_{1+\epsilon} m)$, and we can get the algorithm running in time $n^{O(|\Gamma|)} = n^{O(\log mn)}$.

B.1 Preprocessing

The following lemma says that we can assume the input lies on the grid where each row and column of the grid contains exactly one item.

Lemma B.1. *We are given an instance $(\mathcal{C}, \mathcal{I})$ of 2-UUDP-MIN. Then we can, in polynomial time, transform $(\mathcal{C}, \mathcal{I})$ into an “equivalent” instance $(\mathcal{C}', \mathcal{I}')$ such that*

- Each consumer $\mathbf{C}' \in \mathcal{C}'$ has even coordinates $(2i, 2j)$ for $0 \leq i, j \leq n$.
- Each item $\mathbf{I}' \in \mathcal{I}'$ has odd coordinate $(2i - 1, 2j - 1)$ for $1 \leq i, j \leq n$.
- For each odd number $2i - 1$, $1 \leq i \leq n$, there is exactly one item $\mathbf{I}' \in \mathcal{I}'$ with $\mathbf{I}'[1] = 2i - 1$ and exactly one item \mathbf{I}' with $\mathbf{I}'[2] = 2i - 1$.

Proof. We sweep the horizontal line from top to bottom, and whenever the line meets the items $\mathbf{I}'_1, \dots, \mathbf{I}'_z$ such that $\mathbf{I}'_1[1] < \mathbf{I}'_2[1] < \dots < \mathbf{I}'_z[1]$ with the same y -coordinate y' , we break ties as follows. Let δ be the vertical distance from the line to the next item point below the line. We set the new y -coordinates of these items to $\mathbf{I}'_j[2] = y' - (z - j)\delta/2z$. Notice that some consumers whose y -coordinates lie in $[y', y' - \delta)$ get affected by this move. We also change the y -coordinates of those consumers to $y' - \delta/2$. Then we add the horizontal grid lines between the space of every consecutive items, while making sure that consumer points are on the line passing $y - \delta/2$. It is easy to see that this process preserves the consideration set of every consumer. We repeat the above steps until the sweeping line passes the bottommost item.

We do a similar sweep of vertical line from right to left, inserting the grid lines along the way. In the end, each consumer lies on the intersection of the grid lines and each item in its cell, which guarantees that no two items appear in the same row or column of the grid. \square

B.2 Detail of QPTAS for UUDP-MIN

First, we can make the following simple assumption.

Lemma B.2. *We can assume that the prices are in the form $(1 + \epsilon)^0, (1 + \epsilon)^1, \dots, (1 + \epsilon)^q$ or zero where $q = O(\log_{1+\epsilon} m)$ by sacrificing $(1 + \epsilon)$ in the approximation factor.*

Proof. We use the following standard arguments. Consider an optimal price p^* . For each item \mathbf{I}_j , if the price is non-zero, we round down the price $p^*(\mathbf{I}_j)$ to the nearest scale of $(1 + \epsilon)^{q'}$, so the price of each item gets decreased by at most a factor of $(1 + \epsilon)$. Consider a consumer \mathbf{C} who bought \mathbf{I}_j with price p^* . After the rounding, she can still afford \mathbf{I}_j , so we can still collect at least $(1 - \epsilon)p^*(\mathbf{I}_j)$ from \mathbf{C} . \square

Now, assuming that the optimal price p^* has the above structure, we show how to solve the problem in quasi-polynomial time. First, we reorder the items based on their y -coordinates in descending order, so we have $\mathbf{I}_1[2] > \mathbf{I}_2[2] > \dots > \mathbf{I}_n[2]$. A consumer \mathbf{C} is said to belong to *level* j if it lies between the row of \mathbf{I}_j and that of \mathbf{I}_{j+1} , so each consumer belongs to exactly one level. Moreover, observe that a consumer \mathbf{C} at level j is only interested in (a subset of) items in $\{\mathbf{I}_1, \dots, \mathbf{I}_j\}$ (since $\mathbf{I}_{j'}[2] < \mathbf{C}[2]$ for any $j' > j$). We define a subproblem \mathcal{P}_j as the pricing problem with items $\{\mathbf{I}_1, \dots, \mathbf{I}_j\}$ and consumers at levels $1, \dots, j$. We use the dynamic programming technique to solve this problem.

Profiles We will remember the profile for each subproblem \mathcal{P}_j . A profile Π of \mathcal{P}_j consists of $O(\log m)$ item indices $\pi_1, \dots, \pi_q \in \{1, \dots, j\}$. Each value π_i is supposed to tell us the index of the item \mathbf{I} of price $(1 + \epsilon)^i$ with maximum value $\mathbf{I}[1]$. That is, we say that a price p for \mathcal{P}_j is *consistent* with profile $\Pi = (\pi_1, \dots, \pi_q)$ if, for each i , the item \mathbf{I}_{π_i} has the highest value in the first coordinate among the items with price at most $(1 + \epsilon)^i$, i.e., for all i ,

$$\pi_i = \arg \max_{j'} \{\mathbf{I}_{j'}[1] \mid p(\mathbf{I}_{j'}) \leq (1 + \epsilon)^i\}.$$

Since $\{\mathbf{I}_{j'} \mid p(\mathbf{I}_{j'}) \leq (1 + \epsilon)^i\} \subseteq \{\mathbf{I}_{j'} \mid p(\mathbf{I}_{j'}) \leq (1 + \epsilon)^{i+1}\}$ for any i ,

$$\mathbf{I}_{\pi_1}[1] \leq \mathbf{I}_{\pi_2}[1] \leq \dots \leq \mathbf{I}_{\pi_q}[1].$$

Observe that if two prices p' and p'' have the same \mathcal{P}_j profile, then consumers at level j see no difference between these two prices, as shown formally by the following lemma. We say that an item \mathbf{I}_k is a profile item for profile $\Pi = (\pi_1, \dots, \pi_q)$ if and only if $k = \pi_{q'}$ for some $q' \in [q]$.

Lemma B.3. *Let Π be a profile for subproblem \mathcal{P}_j , and let p be any price function for \mathcal{P}_j that is consistent with profile Π . Then we can assume without loss of generality that every consumer at level j only purchases profile items.*

Proof. Suppose that a consumer \mathbf{C} buys an item \mathbf{I} in \mathcal{I} with $p(\mathbf{I}) = (1 + \epsilon)^{q'}$ which is not a profile item. Then consider the profile item $\mathbf{I}_{\pi_{q'}}$, which satisfies $\mathbf{I}[1] \geq \mathbf{I}_{\pi_{q'}}[1]$, so we must have $\mathbf{I}_{\pi_{q'}} \in S_{\mathbf{C}}$. We can therefore assume that consumer \mathbf{C} buys $\mathbf{I}_{\pi_{q'}}$ instead of \mathbf{I} . \square

Let $\Pi = (\pi_1, \dots, \pi_q)$ be a profile for \mathcal{P}_j and $\Pi' = (\pi'_1, \dots, \pi'_q)$ be a profile for \mathcal{P}_{j-1} . We say that Π is *consistent* with Π' if for any price $p' : \{\mathbf{I}_1, \dots, \mathbf{I}_{j-1}\} \rightarrow \mathbb{R}$ that is consistent with Π' , we can extend p' to p by assigning value $p(\mathbf{I}_j)$ such that p is consistent with Π . Notice that consistency between any two profiles can be checked in polynomial time by trying all q possibilities of prices.

We recall that we use p^* to denote the optimal price.

Lemma B.4. *There are profiles Π^1, \dots, Π^n for $\mathcal{P}_1, \dots, \mathcal{P}_n$ respectively such that for each $j \in \{1, \dots, n-1\}$, Π^j is consistent with Π^{j+1} . Moreover, all such profiles are consistent with price p^* .*

Proof. For each subproblem \mathcal{P}_j , we define the profile $\Pi^j = (\pi_1^j, \dots, \pi_q^j)$ based on the price p^* (there is only one possible profile consistent with p^*). It is clear that Π^j is always consistent with Π^{j+1} . \square

Dynamic Programming Table For each $j = 1, \dots, n$ and for each profile Π of \mathcal{P}_j , we use a table entry $T(j, \Pi)$ to store the maximum revenue achievable among the price function for \mathcal{P}_j that is consistent with the profile Π . Since there are $n^{O(\log m)}$ possibilities for the profile Π , the table size is $n^{O(\log m)}$. We now show the computation of the table. To compute $T(j, \Pi)$, we recall that given the profile Π , the revenue from consumers at level j can be computed efficiently. Denote such revenue by $r_j(\Pi)$. The following equation holds:

$$T(j, \Pi) = r_j(\Pi) + \max_{\Pi' \text{ consistent with } \Pi} T(j-1, \Pi')$$

Computing the Solution For each table entry $T(j, \Pi)$, we can keep track of the profile Π' such that $T(j-1, \Pi')$ is the entry that is used to compute $T(j, \Pi)$. Let $T(n, \Pi)$ be the entry that contains the maximum value over all Π . The value in this entry represents the revenue we can get from the optimal pricing p^* , so it is enough to reconstruct the price function p^* . We first obtain a sequence of profiles $\Pi^1, \dots, \Pi^n = \Pi$ such that Π^j is a profile for \mathcal{P}_j and that Π^j is consistent with Π^{j-1} for any $j = 1, \dots, n$. This sequence allows us to reconstruct a price function that is consistent with all the profiles in polynomial time.

C Omitted hardness results

C.1 Hardness of 3-UUDP-MIN and 4-UUDP-MIN

In this section we show that 3-UUDP-MIN is NP-hard, and 4-UUDP-MIN is APX-hard by a reduction from Vertex Cover. Our reduction relies on the concepts of adjacency poset and its embedding into

Euclidean space. We describe basic terminologies here. Given a graph $G = (V, E)$, an adjacency poset $(V \cup E, \preceq_G)$ of graph G can be constructed as follows: First we define a poset with its maximal elements corresponding to vertices in V and its minimal elements corresponding to edges E . For each vertex v and each edge e , we have the relation $e \preceq_G v$ if and only if vertex v is an endpoint of e . We say that a map $\varphi : V \cup E \rightarrow \mathbb{R}^d$ is an *embedding* of adjacency poset $(V \cup E, \preceq_G)$ into \mathbb{R}^d if and only if it preserves the relations \preceq_G , i.e., for any two elements $a, b \in V \cup E$, we have that $a \preceq_G b$ iff $\varphi(a)[i] \leq \varphi(b)[i]$ for all coordinates $i \in [d]$.

Now we describe our reductions. Since two reductions are essentially the same, we give a general procedure which will imply both results. Given an instance $G = (V, E)$ of **Vertex Cover**, we first construct an adjacency poset $(V \cup E, \preceq_G)$ for G , and then we compute the embedding φ of this poset into Euclidean space \mathbb{R}^d . We will use the graph G , as well as the embedding φ , to define the instance of d -UUDP-MIN as follows:

- **Consumers:** We have two types of consumers, i.e. the rich consumers and the poor ones. For each vertex $v \in V$, we create a *rich* consumer C_v with budget 2 at coordinates $\varphi(v)$. For each edge $e \in E$, we create a *poor* consumer C_e with budget 1 at coordinates $\varphi(e)$.
- **Items:** For each vertex $v \in V$, we create item \mathbf{I}_v at coordinates $\varphi(v)$.

Note that for each $e = (u, v)$, each poor consumer C_e has $S_{C_e} = \{\mathbf{I}_v, \mathbf{I}_u\}$, while each rich consumer C_v has $S_{C_v} = \{\mathbf{I}_v\}$. We denote the resulting instance by $(\mathcal{C}, \mathcal{I})$.

The following lemma gives a characterization of the optimal solution for $(\mathcal{C}, \mathcal{I})$. It says that we may assume without loss of generality that every poor consumer gets some item.

Lemma C.1. *For any price p that is a solution for $(\mathcal{C}, \mathcal{I})$ constructed above, we can transform p to p' such that every poor consumer buys some item with respect to p' , and the revenue of p' is at least as much as the revenue of p .*

Proof. Consider edge $e = (u, v)$. Suppose poor consumer C_e does not get any item, so it implies that both items \mathbf{I}_u and \mathbf{I}_v have price $p(\mathbf{I}_u) = p(\mathbf{I}_v) = 2$ (recall that, since budgets are 1 or 2, the optimal prices would never set prices that are not in $\{1, 2\}$). We define the price function p' by setting $p'(\mathbf{I}_u) = 1$ while $p'(\mathbf{I}_w) = p(\mathbf{I}_w)$ for all other vertices $w \in V \setminus \{u\}$. The only rich consumer that gets affected is C_u , whose payment may decrease by one. However, we earn the revenue of one back from poor consumer C_e . For $e' \in E : e' \neq e$, poor consumer $C_{e'}$ is never affected because his budget is one. Overall, changing the price from p to p' never decreases revenue. \square

Let p^* be the optimal price for $(\mathcal{C}, \mathcal{I})$ and $\text{VC}(G)$ denote the size of minimum vertex cover of G . We show the following connection between the size of minimum vertex cover and the optimal revenue collected by p^* .

Theorem C.2. *The optimal revenue collected by p^* is exactly $2n - \text{VC}(G) + m$.*

Proof. From the previous lemma, we can assume that the pricing p^* sells items to every poor consumer. In other words, if $V' = \{v : p^*(\mathbf{I}_v) = 1\}$, it must be the case that V' is a vertex cover: otherwise, let $e = (u, w)$ be an edge which is not covered by any vertex in V' , so C_e is only interested in items with price 2, which he cannot afford. This contradicts the assumption that p^* sells items to every poor consumer.

The revenue collected from poor consumers is exactly m . Each rich consumer C_v in the vertex cover gets the item with price 1 while others get the items with price 2, so the total revenue is $m + \text{VC}(G) + 2(n - \text{VC}(G))$. \square

This theorem immediately implies the gap between YES-INSTANCE and NO-INSTANCE for d -UUDP-MIN. The only detail we left out is the computation of the embedding φ , and this is where the hardness proofs of 3-UUDP-MIN and 4-UUDP-MIN depart (other steps are exactly the same). For 3-dimensional case, we start from planar graphs whose adjacency poset can be embedded into \mathbb{R}^3 . Since planar vertex cover has a polynomial-time approximation scheme, we only get NP-hardness here. For 4-dimensional case, we start from vertex cover in cubic graphs, which is known to be APX-hard, but unfortunately we can only embed its adjacency poset into \mathbb{R}^4 , thus obtaining the hardness of 4-UUDP-MIN.

NP-Hardness of 3-UUDP-MIN To show the NP-hardness, we start from Vertex Cover in planar graphs, which is known to be NP-complete [31]. We will use the following theorem, due to Schnyder [49].

Theorem C.3. *Let $(V \cup E, \preceq_G)$ be an incident poset of planar graph G . Then there exists an embedding φ from the poset into \mathbb{R}^3 .*

Schnyder shows later that the crucial step in the theorem can be computed in polynomial time [50], which immediately implies the following theorem.

Theorem C.4. *3-UUDP-MIN is NP-hard even when the consumer budgets are either 1 or 2.*

APX-Hardness of 4-UUDP-MIN We will be using the fact that Vertex Cover in cubic graphs is APX-hard [3], stated in the language convenient for our use below.

Theorem C.5. *For some $0 < \alpha < \beta < 1$, it is NP-hard to distinguish between (i) the graph that has a vertex cover of size at most αn , and (ii) the graph whose minimum vertex cover is at least βn .*

Now we assume that our input graph G is a cubic graph and use the following theorem to embed the adjacency poset of G into \mathbb{R}^4 .

Theorem C.6 (Schnyder). *An adjacency poset of any 4-colorable graph can be embedded into \mathbb{R}^4 . Moreover, the embedding is computable in polynomial time.*

It only requires a straightforward computation to prove the following theorem.

Theorem C.7. *4-UUDP-MIN is APX-hard even when the consumer budgets are either 1 or 2.*

Proof. In the YES-INSTANCE, we can collect the revenue of $(2 - \alpha)n + m$. However, in the NO-INSTANCE, the revenue is at most $(2 - \beta)n + m$. Since the graph is cubic, we may assume that $m = \gamma n$ for some $1 \leq \gamma < 2$. Hence we have a gap of $(2 - \alpha + \gamma)/(2 - \beta + \gamma)$. \square

C.2 Hardness Results in Higher Dimensions

In this section, we present the proof of Theorem 3.1. Let $A = (\mathcal{I}, \mathcal{C})$ be an instance of UUDP-MIN where every consumer \mathbf{C} has its consideration set $S_{\mathbf{C}}$ of size at most B . Let $\mathcal{I} = \{\mathbf{I}_1, \dots, \mathbf{I}_n\}$. For each $i \in [d]$, we pick a random permutation $\pi_i : [n] \rightarrow [n]$, so we have d permutations π_1, \dots, π_d . The function φ on items \mathcal{I} can be defined as $\varphi(\mathbf{I}_j)[i] = \pi_i(j)$, and we extend the function to the set of consumers as follows: $\varphi(\mathbf{C})[i] = \min_{j \in S_{\mathbf{C}}} \pi_i(j)$. Now we have a well-defined function φ .

Lemma C.8. *With probability at least $1 - 1/n$, for all consumer $\mathbf{C} \in \mathcal{C}$, the consideration set $S'_\mathbf{C}$ defined by $S'_\mathbf{C} = \{\mathbf{I}_j : \varphi(\mathbf{I}_j) \text{ dominates } \varphi(\mathbf{C})\}$ is exactly $S_\mathbf{C}$.*

Proof. Since we define $\varphi(\mathbf{C})$ to be the minimum of $\varphi(\mathbf{I}_j)$ over all items in $S_\mathbf{C}$, we have $S_\mathbf{C} \subseteq S'_\mathbf{C}$. Let k be the index of an item that does not belong to $S_\mathbf{C}$. We show the following claim.

Claim C.9. *The probability that $\varphi(\mathbf{I}_k)$ dominates $\varphi(\mathbf{C})$ is at most $1/n^{B+2}$.*

Proof. Fix some $i \in [d]$. The bad event that $\pi_i(k) \geq \min_{j \in S_\mathbf{C}} \pi_i(j)$ happens only if π_i does not put k in the last position among $S_\mathbf{C} \cup \{k\}$. This probability is exactly $(1 - 1/(B+1))$. Therefore, the bad event happens for all values of i with probability at most $(1 - 1/(B+1))^d \leq 1/n^{B+2}$ for $d = O(B^2 \log n)$. \square

This claim immediately implies the lemma: By the union bound, the probability that $\varphi(\mathbf{I}_k)$ dominates $\varphi(\mathbf{C})$ is at most $1/n^{B+1}$. So we have that $\Pr[S_\mathbf{C} \neq S'_\mathbf{C}] \leq 1/n^{B+2}$. There are at most n^B possible consideration sets of size at most B , so by union bounds, the probability that a bad event $S_\mathbf{C} \neq S'_\mathbf{C}$ happens for some consumer \mathbf{C} is at most $1/n$. \square

n attributes capture general problem Finally, we end this section with the proof that n -UUDP-MIN captures the whole generality of UUDP-MIN: Consider an instance $(\mathcal{C}, \mathcal{I}, \{S_\mathbf{C}\}_{\mathbf{C} \in \mathcal{C}})$ of UUDP-MIN. Denote the set of items by $\mathcal{I} = \{\mathbf{I}_1, \dots, \mathbf{I}_n\}$. Notice that we can define the coordinates of each consumer by $\mathbf{C}[i] = 0$ if $\mathbf{I}_i \in S_\mathbf{C}$, and $\mathbf{C}[i] = 1$ otherwise. We define the coordinates of each item as $\mathbf{I}_i[i] = 0$ and $\mathbf{I}_i[j] = 1$ for all $j \neq i$. It is easy to check that the consideration sets are preserved by this reduction.